

Representation of quantum field theory in an extended spin space and fermion mass hierarchy

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Abstract

We consider a matrix space based on the spin degree of freedom, describing both a Hilbert state space, and its corresponding symmetry operators. Under the requirement that the Lorentz symmetry be kept, at given dimension, scalar symmetries, and their representations are determined. Symmetries are flavor or gauge-like, with fixed chirality. After spin 0, 1/2, and 1 fields are obtained in this space, we construct associated interactive gauge-invariant renormalizable terms, showing their equivalence to a Lagrangian formulation, using as example the previously studied (5+1)-dimensional case, with many standard-model connections. At 7+1 dimensions, a pair of Higgs-like scalar Lagrangian is obtained naturally producing mass hierarchy within a fermion flavor doublet.

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1. Introduction

The current theory of elementary particles, the standard model (SM), is successful in describing their behavior, but it is phenomenological. The origin of the interaction groups, the particles' spectrum and representations, and parameters has remained largely unexplained. A unified theory can aim to build physical objects from the most elementary ones. The generalization of features of the model into larger structures with a unifying principle has suggested connections among the observables. Thus, additional spatial dimensions in Kaluza-Klein theories are associated with gauge symmetries, and larger gauge groups in grand-unified theories (GUTs)[1] put some restrictions on them.

Particles and interactions obey Lorentz-scalar and local symmetries, associated to gauge groups. The fundamental representation of the Lorentz group is physically manifested in elementary-particle fermions, while the vector representation corresponds to interaction bosons. On the other hand, fermions occupy the scalar-group fundamental representation, and vector particles the adjoint. In addition, the description and quantification¹ of particles and interactions have similar consistency requirements, as restrictions on the representations from unitarity. These notable similarities and connections between the existing particles' discrete degrees of freedom point to a common origin, and hence, a simple composite description.

Indeed, a shared vector space, was proposed[2, 3] that generalizes spin, and accommodates scalar and Lorentz degrees of freedom; at given dimension[d], this space constrains the symmetries and representations, and its generators in the dimensions beyond 3+1 are associated with scalar symmetries. While only a simple admixture of Lorentz and scalar groups is permitted by the Coleman-Mandula theorem[4], additional non-trivial information is obtained from the spin-space scheme, as chiral and vector characterizations emerge from the symmetries and particle representations.

¹Expressed in quantum numbers within quantum mechanics.

Similarly to the supersymmetry case[6], the dimension of the space constrains the particle spectrum; as we will explain, the interactions are also constrained.

Within a Kaluza-Klein framework, this extension may be viewed as a consequence of the spatial components being frozen. Conceptually, the matrix construction stems from incremental direct products with 2×2 matrices, suggesting the discrete Hilbert space used is built up from the most elementary degrees of freedom (e. g., q-bits or spin-1/2 particles.)

Although standard SM extensions provide additional information on it, many puzzles remain unsolved. With its bottom-up approach, this model reduces the available groups and representations to fit particles and their quantum numbers, in contrast, e. g., to the representation choices available in GUTs, and to the multiplicity of compactification options that plagues strings.

While this scheme was used before to derive information on coupling constants[7], SM representations [7, 8], and relations between electroweak boson masses[8], a formal treatment to produce an interactive model was missing. In this paper, we construct step-by-step gauge- and Lorentz-invariant terms from fields within representations and symmetries that derive from the extended spin space, which translates into a Poincaré-invariant Lagrangian theory. In particular, we show formally the equivalence of a gauge-invariant field theory, written in such a space, and a standard formulation, thus extending and complementing previous work[7, 8, 9]; each vertex type exhibits particular features. We also find that the scalar fermion-scalar term in (7+1)-d implies a hierarchy in the fermion masses.

The paper is organized as follows. In Section 2, we review the spin-space extension, based on the Dirac equation, and its connection to a matrix space. In particular, we present its elements' classification, using a Clifford algebra, under the demand that Lorentz symmetry be maintained; in such a space, spinors belong to the scalar-group fundamental representation, while vectors to the adjoint representation[2, 3, 7, 8, 9].

In Section 3, generalized fields and symmetries are expressed in this space (using as example the (5+1)-d case). In Section 4, these are used to construct a gauge-invariant interactive theory, showing that it can be formulated in terms of a standard Lagrangian; we deal with vector-scalar, fermion-vector and vector-vector vertices, using the obtained groups $SU(2)_L \otimes U(1)_Y$ in (5+1)-d, and correct chirality. In Section 5, fermion-scalar vertices are obtained in 7+1 d. Higgs-like scalars emerge that lead, through the Higgs mechanism, to fermion masses in a flavor doublet; Yukawa couplings naturally generate a fermion-mass hierarchy. In Section 6, we summarize relevant points in the paper.

Other investigations similarly rely on the spin degree of freedom in SM extensions[10, 11], in trying to understand its still unresolved questions. These use an algebraic spinor represented by a matrix, where the common feature of this type of model building is the use of the structure within an associated Clifford-algebra space. In four dimensions, a 4×4 matrix connects to the $(3 + 1)$ -d Clifford algebra \mathcal{C}_4 . Each column in the matrix is a left ideal of the algebra. This allows for operators acting from the right, and such transformations are usually associated with gauge groups. To take account of the SM particle multiplets and gauge groups, one introduces extra spacetime dimensions. Different choices are made for the nature of the left ideals, the spacetime dimensions, and symmetry transformations, which leads to different models with various degrees of applicability and phenomenological implications. In Refs. [12, 13] models based on Clifford objects in $13 + 1$ d purport to explain the origin of quark and lepton families. In Ref. [14] an algebraic spinor of \mathcal{C}_7 is used to represent one family of quarks and leptons, with Poincaré and gauge transformations restricted to act from the left and right, respectively.

Other types of models include gravity and are geometric in nature. Thus, the fundamental Clifford algebra relation, usually taken as a real algebra, is given in

terms of an abstract vector basis $\{e_\mu\}$ as

$$e_\mu e_\nu + e_\nu e_\mu = g_{\mu\nu}, \quad (1)$$

without reference to the gamma matrices. To cite some recent examples (in no manner an exhaustive list), in Refs. [15] and [16] models include the SM gauge groups and gravity, the former based on \mathcal{C}_6 , and the latter on \mathcal{C}_{3+1} , which assumes a column spinor within an algebraic matrix. Ref. [17] also advances a model including gauge and gravity fields, motivated by strings and branes models, and set up in a 16-d Clifford space.

2. Gamma-matrix symmetry classification

The Dirac equation formulated over the matrix Ψ (and corresponding conjugate equation)

$$\gamma_0(i\partial_\mu\gamma^\mu - M)\Psi = 0, \quad (2)$$

may be used as framework for the classification of states and operators in an extended space,² and study symmetry transformations. It also generates free-particle fermion and bosons on the extended space.

These matrices generate an algebra, and may be also viewed in terms of their bra-ket components:

$$\Gamma = \sum c_{ab}|a\rangle\langle b|, \quad (3)$$

with c_{ab} c -numbers. The dot product between the elements Γ_a , Γ_b can be defined using the trace

$$\text{tr } \Gamma_a^\dagger \Gamma_b. \quad (4)$$

²We assume throughout $\hbar = c = 1$, and 4-d diagonal metric elements $g_{\mu\nu} = (1, -1, -1, -1)$.

Assuming Γ in Eq. 3 to be unitary we obtain the condition [2, 3] on the c_{ab} values

$$\Gamma^\dagger \Gamma = \sum c_{ba}^* c_{bc} = \delta_{ac}. \quad (5)$$

Implicit in the $|a\rangle\langle b|$ matrix construction is the appropriate transformation operators U acting on field states Ψ ; these can generically be characterized by the expression

$$\Psi \rightarrow U\Psi U^\dagger. \quad (6)$$

We show next that a matrix Γ can be associated to either Ψ and U , the latter representing both Lorentz and scalar symmetries. We also show that a 4-dimensional Clifford matrix subalgebra is obtained, implying spinor up to bi-spinor elements, thus vectors and scalar fields, can be described.

An operator Op within this space characterizes a state Ψ with the eigenvalue rule

$$[Op, \Psi] = \lambda\Psi, \quad (7)$$

consistent with the hole interpretation, and anticipating a second-quantization description. For example, an on-shell boson may be constructed by two fermion components with positive frequencies $\psi_1(x)$, $\bar{\psi}_2(x)$ through $\psi_1(x)\bar{\psi}_2(x)$, following Eq. 3, with $\bar{\psi}_2(x)$ describing an antiparticle.

If Eq. 2, keeping $\mu = 0, \dots, 3$, is assumed within the larger Clifford algebra³ \mathcal{C}_N , $\{\gamma_\eta, \gamma_\sigma\} = 2g_{\eta\sigma}$, $\eta, \sigma = 0, \dots, N-1$, with N the (assumed even) dimension, whose structure is helpful in classifying the available symmetries U , and solutions Ψ , both represented by $2^{N/2} \times 2^{N/2}$ matrices. The 4-d Lorentz symmetry is maintained, and uses the generators

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu], \quad (8)$$

where $\mu, \nu = 0, \dots, 3$. U contains also γ_a , $a = 4, \dots, N-1$, and their products as possible symmetry generators. The $N = 4$ case was analyzed in Ref. [2, 3], $N = 6$ in [2], [3],

³Understood here also as a matrix space.

[8], and $N = 10$ in [7]. Indeed, the latter elements are scalars for they commute with the Poincaré generators, which contain $\sigma_{\mu\nu}$, and they are also symmetry operators of the massless Eq. 2, bilinear in γ_μ , $\mu = 0, \dots, 3$ which is not necessarily the case for mass terms (containing γ_0). In addition, their products with $\tilde{\gamma}_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$ are Lorentz pseudoscalars. As $[\tilde{\gamma}_5, \gamma_a] = 0$, we can classify the (unitary) symmetry algebra as $\mathcal{S}_{N-4} = \mathcal{S}_{(N-4)R} \otimes \mathcal{S}_{(N-4)L}$, consisting of the projected right-handed $\mathcal{S}_{(N-4)R} = \frac{1}{2}(1 + \tilde{\gamma}_5)U(2^{(N-4)/2})$ and left-handed $\mathcal{S}_{(N-4)L} = \frac{1}{2}(1 - \tilde{\gamma}_5)U(2^{(N-4)/2})$ components, where $U(M)$ is a representation of the M -unitary group in \mathcal{C}_N . Its reduced form $\tilde{U}(2^{(N-4)/2}) \subset \mathcal{C}_{N-4}$, with $\mathcal{C}_N = \mathcal{C}_4 \otimes \mathcal{C}_{N-4}$, is the irreducible fundamental representation. The operator algebra was described in Refs. [7] and [9].

A state Ψ is classified in accordance with the above symmetry generators that emerge from the Clifford algebra. For given dimension N , any matrix element representing a state is obtained by combinations of products of one or two γ_μ , and elements of \mathcal{S}_{N-4} , which define, respectively, their Lorentz (as for 4-d) and scalar-group representation. There is a finite number of partitions on the matrix space for the states and symmetry operators, consistent with Lorentz symmetry. The variations of the symmetry algebra are defined by the projection operators \mathcal{P}_P , $\mathcal{P}_S \in \mathcal{S}_{N-4}$ with $[\mathcal{P}_P, \mathcal{P}_S] = 0$; \mathcal{P}_P acts on the Lorentz generator

$$\mathcal{P}_P \left[\frac{1}{2} \sigma_{\mu\nu} + i(x_\mu \partial_\nu - x_\nu \partial_\mu) \right], \quad (9)$$

and \mathcal{P}_S on the symmetry operator space

$$\mathcal{S}'_{N-4} = \mathcal{P}_S \mathcal{S}_{N-4}, \quad (10)$$

leading to projected scalar generators $I_a = \mathcal{P}_S I_a$, so that they determine, respectively, the Poincaré generators and the scalar groups.

(*)		
	$\mathcal{S}_{(N-4)R} \otimes \mathcal{C}_4$	
		$\mathcal{S}_{(N-4)L} \otimes \mathcal{C}_4$

1(a)

The application of these operators follows the operator rule in Eq. 7, which assigns states to particular Lorentz and scalar group representations. For simplicity, we assume $\mathcal{P}_P = \mathcal{P}_S \neq 1$, as other possibilities are less plausible[7]. Thus, the Lorentz or scalar operators act trivially on one side of solutions of the form $\Psi = \mathcal{P}_P \Psi (1 - \mathcal{P}_P)$, since $(1 - \mathcal{P}_P) \mathcal{P}_P = 0$, leading to spin-1/2 states or states belonging to the fundamental representation of the non-Abelian symmetry groups, respectively.

On Table 1(a), we show schematically the organization of the symmetry operators, producing corresponding Lorentz and scalar generators. Table 1(b) also depicts the resulting solution representations, distributed according to their Lorentz classification: fermion, scalar, vector, and antisymmetric tensor. The matrices are classified according to the chiral projection operators $\frac{1}{2}(1 \pm \tilde{\gamma}_5)$, leading to $N/2 \times N/2$ matrix blocks in \mathcal{C}_N . The space projected by $\mathcal{P}_P = \mathcal{P}_S \neq 1$ is also depicted.

The chiral property of the fermion representations contrasts with the difficulty to reproduce it in traditional Kaluza-Klein extensions[18]. In addition, when deriving a

(*)	F	F
F	V	S,A
F	S,A	V

1(b)

Table 1: (a) shows the arrangement of symmetry operators U in matrix space of arbitrary dimension N , after projection over \mathcal{S}_P , with left-handed and right-handed operators subspaces[7]; (*) represents the matrix subspace containing the projector $1 - \mathcal{P}_S = 1 - \mathcal{P}_P$; its choice within the right-handed symmetry components is arbitrary. (b) shows the arrangement of matrix solutions Ψ in the extended-spin model is divided into four $\frac{N}{2} \times \frac{N}{2}$ matrix blocks, containing fermion (F), vector (and axial-) (V), and scalar (and pseudo-), and antisymmetric (S,A) terms.

unitary subgroup $SU(M - 4)$, for arbitrary M , departing from an extended Lorentz group requires $O(2M - 5, 1) \supset SU(M - 4) \otimes O(3, 1)$, while in our scheme, the subgroup chain can be chosen as $U(M - 4) \supset O(3, 1) \otimes SU(M - 4)_R \otimes SU(M - 4)_L$. This means lower dimensional spaces are sufficient to reproduce the SM groups, reducing the representation sizes, and eliminating spurious degrees of freedom; in addition, the right- and left-handed group separation is possible for all dimensions.

While a grand-unified group limits the representations among which one must choose to put particles, in our case, the representations are determined. Indeed, the specific combinations also emerge, corresponding to spin-1/2-fundamental and vector-

adjoint, Lorentz and scalar groups representations, respectively; graphically, vectors and scalar group elements occupy the same matrix spots (and similarly for fermions,) as seen in Tables 1(a) and 1(b).

In the next Section, we generalize these fields.

3. Fields and symmetries in matrix space

To construct interactive fields, we start with free fields within the (5+1)-d space[8] case as example, for which we highlight predicted physical features. There, among few choices, $\mathcal{P}_P = L$, with $L = \frac{3}{4} - \frac{i}{4}(1 + \tilde{\gamma}_5)\gamma^5\gamma^6 - \frac{1}{4}\tilde{\gamma}_5$ is associated to the lepton number, and the resulting symmetry generators and particle spectrum fits the SM electroweak sector. Specifically, the projected symmetry space also includes the $SU(2)_L \otimes U(1)_Y$ groups, with respective generators I_i and hypercharge Y

$$\begin{aligned}
I_1 &= \frac{i}{4}(1 - \tilde{\gamma}_5)\gamma^5 \\
I_2 &= -\frac{i}{4}(1 - \tilde{\gamma}_5)\gamma^6 \\
I_3 &= -\frac{i}{4}(1 - \tilde{\gamma}_5)\gamma^5\gamma^6 \\
Y &= -1 + \frac{i}{2}(1 + \tilde{\gamma}_5)\gamma^5\gamma^6.
\end{aligned} \tag{11}$$

We note that the $SU(2)$ generators correctly contain the projection operator $\frac{1}{2}(1 - \tilde{\gamma}_5)$, confirming the interaction's chiral nature, which also leads to chiral representations, a feature that results from nature of the matrix space under projector L and the Lorentz group. Under Eq. 7, the action of these operators on choices of free-particle states Ψ is given on Table 2 together with their quantum numbers.

The question on what fixes this extension's dimension to derive SM groups and representations similarly applies to GUTs, as there is also an infinite number of pos-

sible groups that contain the SM. The answers for both extensions hinge on that the lowest dimension numbers already give relevant information, and on predictability as, in our case, features as representations and chiral SU(2) are derived.

Electroweak multiplets	States Ψ	I_3	Y	Q	L	$\frac{i}{2}L\gamma^1\gamma^2$	$L\tilde{\gamma}_5$
Fermion doublet	$\frac{1}{8}(1 - \tilde{\gamma}_5)(\gamma^0 + \gamma^3)(\gamma^5 - i\gamma^6)$	1/2	-1	0	1	1/2	-1
	$\frac{1}{8}(1 - \tilde{\gamma}_5)(\gamma^0 + \gamma^3)(1 + i\gamma^5\gamma^6)$	-1/2	-1	-1	1	1/2	-1
Fermion singlet	$\frac{1}{8}(1 + \tilde{\gamma}_5)\gamma^0(\gamma^0 + \gamma^3)(\gamma^5 - i\gamma^6)$	0	-2	-1	1	1/2	1
Scalar doublet	$\frac{1}{4\sqrt{2}}(1 - \tilde{\gamma}_5)\gamma^0(1 - i\gamma^5\gamma^6)$	1/2	1	1	0	0	-2
	$\frac{1}{4\sqrt{2}}(1 - \tilde{\gamma}_5)\gamma^0(\gamma^5 + i\gamma^6)$	-1/2	1	0	0	0	-2
Vector triplet	$\frac{1}{4}(1 - \tilde{\gamma}_5)\gamma^0(\gamma^1 + i\gamma^2)(\gamma^5 - i\gamma^6)$	1	0	1	0	1	0
	$\frac{1}{2\sqrt{2}}(1 - \tilde{\gamma}_5)\gamma^0(\gamma^1 + i\gamma^2)\gamma^5\gamma^6$	0	0	0	0	1	0
	$\frac{1}{4}(1 - \tilde{\gamma}_5)\gamma^0(\gamma^1 + i\gamma^2)(\gamma^5 + i\gamma^6)$	-1	0	-1	0	1	0

Table 2: Massless fermion and bosons states in (5+1)-d extension, momentum along $\pm\hat{\mathbf{z}}$, with projection given by the lepton number $\mathcal{P}_P = L$, under the operators $SU(2)_L$ I_3 component, hypercharge Y , charge $Q = I_3 + \frac{1}{2}Y$, the lepton number L , helicity $\frac{i}{2}L\gamma^1\gamma^2$, and chirality $L\tilde{\gamma}_5$ (the coordinate dependence is omitted.)

By identifying elements between the extended spin space and standard Lagrangian terms, Ref. [8] set thumb rules to derive some gauge-invariant terms. For rigor's sake, and to test the model's reach, it is desirable to obtain such terms within the model's algebra. Next, we translate the field information that emerges from the extended-spin space, to derive an interactive gauge theory. First, we write fields in the extended-spin basis; similarly, the symmetry generators are written in a standard representation; finally, invariant terms are constructed, and shown to be equivalent to field-theory Lagrangian contributions.

As derived in Section 2, and exemplified above, it is possible to write fundamental fields using as basis products of matrices conformed of Lorentz and scalar group representations. Indeed, the commuting property of the respective degrees of freedom allows for states and operators to be written in the form $\mathcal{C}_4 \otimes \mathcal{S}_{N-4}$; explicitly, $\Psi = M_1 M_2$, where

$$M_1 \in \mathcal{C}_4 \quad \text{and} \quad M_2 \in \mathcal{S}_{N-4}. \quad (12)$$

An expression with elements of each set is possible through their passage to each side, using commutation or anticommutation rules.

3.1. Fields' construction

In the presence of interactions, free fields give way to more general expressions of fermion and boson fields, keeping their transformation properties:

Vector field

$$A_\mu^a(x)\gamma_0\gamma_\mu I_a, \quad (13)$$

where $\gamma_0\gamma_\mu \in \mathcal{C}_4$, and $I_a \in \mathcal{S}'_{N-4}$ is a generator of a given unitary group, and \mathcal{S}'_{N-4} is defined in Eq. 10.

Scalar field

$$\phi^a(x)\gamma_0 M_a^S. \quad (14)$$

Fermions

$$\psi_\alpha^a(x)L^\alpha P_F M_a^F, \quad (15)$$

where $M_a^S, M_a^F \in \mathcal{S}_{N-4}$ are, respectively, scalar and fermion components, and L^α represents a spin component; for example, $L^1 = (\gamma_1 + i\gamma_2)$, P_F is a projection operator of the type in Eq. 9, such that

$$P_F\gamma_\mu = \gamma_\mu P_F^c, \quad (16)$$

and we use the complement $P_F^c = 1 - P_F$, so that a Lorentz transformation with $P_F\sigma_{\mu\nu}$, will describe fermions, as argued in Section 3. The simplest example for an operator satisfying such conditions is $P_F = (1 - \tilde{\gamma}_5)/2$ [2, 3], used by the fermion doublet on Table 2. By the argument after Eq. 9, the fundamental-representation state is derived from the trivial right-hand action of the operator within the transformation rule in Eq. 16. The matrix entitles spurious ket states contained in the Lorentz-scalar matrices in Eq. 12.

Antisymmetric tensor

It is also obtained, however, but as it leads to non-renormalizable interactions, it will hence be omitted.

3.2. Symmetry Transformations

We now describe different types of transformations that act as in Eq. 6:

Lorentz Transformation

$$U = \exp\left(-\frac{i}{4}\mathcal{P}_P w^{\mu\nu}\sigma_{\mu\nu}\right), \quad (17)$$

where $\sigma_{\mu\nu}$ is given in Eq. 8, $w^{\mu\nu}$ are parameters and \mathcal{P}_P is the scalar projector in Eq. 9.

Gauge Transformation

$$U = \exp[-iI_a\alpha_a(x)], \quad (18)$$

where $I_a \in \mathcal{S}'_{N-4}$, and $\alpha_a(x)$ are arbitrary functions. The unitary-group representations $\bar{N} \otimes N$, based on elements in the fundamental representation and its conjugate, denoted by N , \bar{N} , respectively, are implicit from the $|a\rangle\langle b|$ matrix construction in Eq. 3; these include the singlet, and the fundamental (expressed in I_a) representations, and similarly those obtained by $N \otimes N$ (see, e.g., [7].)

4. Lagrangian connection

Historically, it is known that Maxwell's equations can be formulated in terms of a Dirac basis[19]. In our case, the fields within the extended-spin basis can be used to construct a standard-formulation Lagrangian.⁴ This amounts to using elements with a well-defined group structure to get Lorentz-scalar gauge-invariant combinations. Choosing scalar elements that result from the direct product in Eq. 4, one obtains an interactive theory, as the same particle content is maintained. In this way, choices of Lagrangians are constrained by the same conditions as in quantum field theory, as renormalizability and quantization.

We proceed by first constructing matrix elements containing the vector field, together with either fermion or bosons fields, and then converting them to expressions

⁴Alternatively, the fields' Lagrangian describing them can be reinterpreted in terms of this basis.

in terms of states' associated bras or kets. Under Lorentz and gauge-group transformations of the extended spin space, invariant elements are obtained by taking the trace. The latter extracts the identity-matrix coefficient, leading to the usual Lagrangian components. The invariance under transformations in Eq. 6 can be verified independently, using the separation in Eq. 12 into Lorentz and scalar symmetries; the invariance will be shown for linear (vector-fermion) or bilinear (vector-scalar) objects, with input from Eqs. 13-15:

4.1. Fermions

A gauge-invariant fermion-vector interaction term results, by adding to the fermion free-term Lagrangian (that implies the Dirac equation 2) the vector-term contribution in Eq. 13

$$\frac{1}{N_f} \text{tr} \Psi^\dagger \{ [i\partial_\mu I_{den} + gA_\mu^a(x)I_a] \gamma_0 \gamma^\mu - M\gamma_0 \} \Psi P_f, \quad (19)$$

where Ψ is a field representing in this case spin-1/2 particles; spin-1 terms are treated below. I_a is the group generator in a given representation, g is the coupling constant, N_f contains the normalization (and similar terms below), and I_{den} the identity scalar group operator in the same representation (which will be omitted hence). An operator P_f is introduced to avoid cancelation of non-diagonal fermion elements. For example,

$$P_f = \frac{1}{\sqrt{2}} [(1+i)(I + \gamma^0 \gamma^2) + \gamma^5 \gamma^6 + \gamma^0 \gamma^2 \gamma^5 \gamma^6] \quad (20)$$

as $[P_f, L] = [P_f, (1 - \tilde{\gamma}_5)L] = 0$, provides a non-trivial combination with the correct quantum numbers for the fermion pair $\Psi_a P_f \Psi_b^\dagger$ (with Ψ_a, Ψ_b either doublet or singlet fermions, on Table 2), and maintains their normalization, spin, lepton and electroweak representation.

As explained after Eq. 15, Lorentz and scalar operators act non-trivially only from one side. Given the action of projection operators $\mathcal{P}_S \mathcal{P}_P$, the transformation in Eq. 6 becomes

$$\Psi \rightarrow U\Psi. \quad (21)$$

Eq. 19 is invariant under the Lorentz transformation in Eq. 17, provided the vector field transforms as

$$A_\mu^a(x)I_a \rightarrow \Delta_\mu{}^\nu A_\nu^a(x)I_a, \quad (22)$$

where we use the identity relating the spin representation of the Lorentz group in

$$U\gamma^\mu U^{-1} = (\Delta^{-1})^\mu{}_\nu \gamma^\nu, \quad (23)$$

and $\Delta^\mu{}_\nu$ is a 4×4 Lorentz transformation matrix transforming a coordinate as $x^\mu \rightarrow \Delta^\mu{}_\nu x^\nu$. The equation is also invariant under local transformation in Eq. 18, under the condition the vector field transforms as

$$A_\mu^a(x)I_a \rightarrow UA_\mu^a(x)I_a U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger, \quad (24)$$

The trace in Eq. 19 can be expressed in terms of states, as we rely on the expansion in Eq. 3 for fields Ψ . The fermion field in Eq. 15, with matrix elements $\gamma\delta$, is expressed as $[\psi_\alpha^a(x)L^\alpha P_F M_a^F]_{\gamma\delta} = \sum_\eta (L^\alpha P_F)_{\gamma\eta} (M_a^F)_{\eta\delta} \langle \alpha a x | \Psi \rangle$, where $(L^\alpha P_F)_{\gamma\eta} = \langle \gamma | \alpha \rangle \sum_\beta d_{\alpha\beta} \langle \beta | \eta \rangle$, $(M_a^F)_{\eta\delta} = \langle \eta | a \rangle \sum_b f_{ab} \langle b | \delta \rangle$, with $d_{\alpha\beta}, f_{ab}$ c-numbers, and where the fermion field $\psi_\alpha^a(x) = \langle \alpha a x | \Psi \rangle$ takes account of spin and scalar degrees of freedom. We choose commuting scalar and spin matrices as basis elements, as the fermion singlet on Table 2; we use the separability property[20] of the generators (for anticommuting matrices, as for the doublets, each bilinear term is separated), as the normalization condition, in Eq. 5, cancels the ket-bra: for the spin component, $L_\alpha P_F \tilde{L}_f (L_\beta P_F)^\dagger = (L_\alpha P_F)_{\gamma\delta} (\tilde{L}_f)_{\delta\gamma} (L_\beta P_F)_{\epsilon\gamma}^* = \langle \gamma | \alpha \rangle \langle \beta | \epsilon \rangle$, and similar calculation involving the electroweak states (here \tilde{L}_f is a reduced operator L_f acting on spin degrees of freedom.) This implies Eq. 19 can be written

$$\psi_\beta^{b\dagger}(x) \{ [i\partial_\mu I_{bc} + gA_\mu^a(x)(I_a)_{bc}] (\gamma_0 \gamma^\mu)_{\beta\alpha} - M I_{bc} (\gamma_0)_{\beta\alpha} \} \psi_\alpha^c(x). \quad (25)$$

Thus, Table 2 leads to the fermion electroweak SM Lagrangian contribution[21]-[23], also derived heuristically in Refs. [8] and [9] .

$$\bar{\Psi}_l [i\partial_\mu + \frac{1}{2}g\tau^a W_\mu^a(x) - \frac{1}{2}g'B_\mu(x)]\gamma^\mu \Psi_l + \bar{\psi}_r [i\partial_\mu - g'B_\mu(x)]\gamma^\mu \psi_r, \quad (26)$$

which contains a left-handed hypercharge $Y_l = -1$ SU(2) doublet Ψ_l , and right-handed $Y_r = -2$ singlet ψ_r , and the corresponding gauge-group vector bosons and coupling constants are, respectively, $B_\mu(x)$, $W_\mu^a(x)$, and g, g' .

4.2. Spin-0 Boson

A Lorentz-invariant interaction term between vector and scalar fields is constructed by applying twice the operator contained within the state Ψ in Eq. 19, removing the γ_0 matrix, following the Klein Gordon equation:

$$\text{tr} \frac{1}{N_B} \Psi^\dagger [i\partial_\nu I_{den} + gA_\nu^b(x)I_b] \gamma^\nu \gamma^\mu [i\partial_\mu I_{den} + gA_\mu^a(x)I_a] \Psi, \quad (27)$$

where the transformation in Eq. 6 is now used in the guise $\Psi \rightarrow U\Psi U^{-1}$, and the 4-d γ_μ are positioned in near pairs to maintain the generators I_a relations (see also the vector term in Eq. 28;) this expression applies to the Lorentz transformation as in Eq. 17. The final expression is obtained by applying the equality $\gamma_\mu\gamma_\nu = g_{\mu\nu} - i\sigma_{\mu\nu}$, as the only symmetric term $[i\partial_\nu I_{den} + A_\nu^b(x)I_b][i\partial_\mu I_{den} + A_\mu^a(x)I_a] = \frac{1}{2}\{i\partial_\nu I_{den} + A_\nu^b(x)I_b, i\partial_\mu I_{den} + A_\mu^a(x)I_a\}$ survives the renormalizability demand.⁵ A similar expansion as for the fermion field in Eq. 19 can be performed, the two γ_0 matrices in the field terms in Eq. 14, contained in $\Psi\Psi^\dagger$, lead to the identity matrix within the trace. The vector mass term resulting from the Higgs mechanism was related to mass operators within the spin-extended space, and used to connect it to the SM in Ref. [8].

4.3. Vector Boson

We use invariant components for the vector field contained in Eq. 13 to construct its kinetic-energy term, and we extract the antisymmetric part

$$[i\partial_\nu I_{den} + gA_\nu^b(x)I_b][i\partial_\mu I_{den} + gA_\mu^a(x)I_a]\frac{i}{2}[\gamma^\nu, \gamma^\mu] = F_{\mu\nu}^a I_a \frac{i}{2}[\gamma^\nu, \gamma^\mu], \quad (28)$$

where by taking the antisymmetric tensor $[\gamma^\nu, \gamma^\mu]$ we extract $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g c^{abd} A_\nu^b A_\mu^d$, and c^{abd} are the structure constants of the group $[I_b, I_d] = i c^{abd} I_a$,

We show a particular term that reproduces the kinetic vector contribution, which eliminates non-renormalizable higher-derivative terms. A scalar contribution is constructed from the contraction of the two terms

$$\frac{1}{N_A} \text{tr} F_{\mu\nu}^a I_a \frac{i}{2}[\gamma^\nu, \gamma^\mu] F_{\rho\sigma}^b I_b \frac{i}{2}[\gamma^\rho, \gamma^\sigma]. \quad (29)$$

From the 4-d trace relation

$$\text{tr} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho} \quad (30)$$

the trace is reduced to the anti-symmetrized combination, $-g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}$. We finally get the known expression for the kinetic term $-\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a}$. The expression in Eq. 29 may be also derived from the original corresponding standard Lagrangian[8].

⁵Where we use the operator equality $AB = \frac{1}{2}[A, B] + \frac{1}{2}\{A, B\}$, and the antisymmetric term cancels through the trace on the (3+1)-d spinor indices

5. (7+1)-dimensional electroweak spinors: mass term and hierarchy

We described fermion-vector and vector-scalar Lagrangian contributions, and in this Section we deal with fermion-scalar terms. An inherent aspect of the $(5+1)$ -d space is the impossibility of defining fermion masses for both flavor-doublet components. The $(7+1)$ -d space allows for charge $2/3$ and $-1/3$ terms, associated to quarks, and charge -1 and neutral leptons. We concentrate on quarks, while the results of this section can be equally applied to leptons.

The baryon-number operator $B = \frac{1}{6}(1 - i\gamma_5\gamma_6)$ conforms a spin-space partition obtained with the additional Clifford members γ_7, γ_8 . It allows for quark symmetry generators that include the hypercharge $Y = \frac{1}{6}(1 - i\gamma_5\gamma_6)[1 + i\frac{3}{2}(1 + \tilde{\gamma}_5)\gamma_7\gamma_8]$, the weak $SU(2)_L$ terms

$$\begin{aligned} I_1 &= \frac{i}{8}(1 - \tilde{\gamma}_5)(1 - i\gamma_5\gamma_6)\gamma^7, \\ I_2 &= \frac{i}{8}(1 - \tilde{\gamma}_5)(1 - i\gamma_5\gamma_6)\gamma^8, \\ I_3 &= \frac{i}{8}(1 - \tilde{\gamma}_5)(1 - i\gamma_5\gamma_6)\gamma_7\gamma_8, \end{aligned} \tag{31}$$

flavor generators, and the Lorentz generators, with spin component $3B\sigma_{\mu\nu}$, projected by B , using Eq. 8. As required, $[Y, I_i] = [B, Y] = [B, I_i] = 0$, and all quarks are associated the baryon number $1/3$ ($-1/3$ for antiparticles.)

Examples of quark massless basis states, expressed as in Eq. 15, are summarized on Table 3, for both u and d-type quarks, with their quantum numbers. The spin component along $\hat{\mathbf{z}}$, $i\frac{3}{2}B\gamma^1\gamma^2$, is used. Only one polarization and two flavors are shown, as a more thorough treatment of the fermion flavor states will be given elsewhere[24].

hypercharge 1/3 left-handed doublet	I_3	Q	$\frac{3i}{2}B\gamma^1\gamma^2$
$\mathbf{Q}_L^1 = \begin{pmatrix} U_L^1 \\ D_L^1 \end{pmatrix} = \begin{pmatrix} \frac{1}{16}(1 - \tilde{\gamma}_5)(\gamma^5 - i\gamma^6)(\gamma^7 + i\gamma^8)(\gamma^0 + \gamma^3) \\ \frac{1}{16}(1 - \tilde{\gamma}_5)(\gamma^5 - i\gamma^6)(1 - i\gamma^7\gamma^8)(\gamma^0 + \gamma^3) \end{pmatrix}$	$\begin{matrix} 1/2 \\ -1/2 \end{matrix}$	$\begin{matrix} 2/3 \\ -1/3 \end{matrix}$	$\begin{matrix} 1/2 \\ 1/2 \end{matrix}$

(a)

$I_3 = 0$ right-handed singlet	Y	Q	$\frac{3i}{2}B\gamma^1\gamma^2$
$U_R^1 = \frac{1}{16}(1 + \tilde{\gamma}_5)(\gamma^5 - i\gamma^6)(\gamma^7 + i\gamma^8)\gamma^0(\gamma^0 + \gamma^3)$ $D_R^1 = \frac{1}{16}(1 + \tilde{\gamma}_5)(\gamma^5 - i\gamma^6)(1 - i\gamma^7\gamma^8)\gamma^0(\gamma^0 + \gamma^3)$	$\begin{matrix} 4/3 \\ -2/3 \end{matrix}$	$\begin{matrix} 2/3 \\ -1/3 \end{matrix}$	$\begin{matrix} 1/2 \\ 1/2 \end{matrix}$

(b)

Table 3: (a) Massless left-handed quark weak isospin doublet, and (b) right-handed singlets, with momentum along $\pm \hat{\mathbf{z}}$.

On Table 4 we also present two scalar elements as in Eq. 14, whose quantum numbers associate them to the Higgs doublet. These are unique within the (7+1)-d space[24]. The combination $a\phi_1 + b\phi_2$ for arbitrary real a, b is classified with the chiral

0 baryon-number scalar	I_3	Y	Q	$\frac{3i}{2}B\gamma^1\gamma^2$
$\phi_1 = \begin{pmatrix} \phi_1^+ \\ \phi_1^0 \end{pmatrix} = \begin{pmatrix} \frac{1}{8}(1 - i\gamma_5\gamma_6)(\gamma^7 + i\gamma^8)\gamma^0 \\ \frac{1}{8}(1 - i\gamma_5\gamma_6)(1 + i\gamma_7\gamma_8\tilde{\gamma}_5)\gamma^0 \end{pmatrix}$	$\begin{matrix} 1/2 \\ -1/2 \end{matrix}$	1	$\begin{matrix} 1 \\ 0 \end{matrix}$	0
$\phi_2 = \begin{pmatrix} \phi_2^+ \\ \phi_2^0 \end{pmatrix} = \begin{pmatrix} \frac{1}{8}(1 - i\gamma_5\gamma_6)(\gamma^7 + i\gamma^8)\tilde{\gamma}_5\gamma^0 \\ \frac{i}{8}(1 - i\gamma_5\gamma_6)(1 + i\gamma_7\gamma_8\tilde{\gamma}_5)\gamma^7\gamma^8\gamma^0 \end{pmatrix}$	$\begin{matrix} 1/2 \\ -1/2 \end{matrix}$	1	$\begin{matrix} 1 \\ 0 \end{matrix}$	0

Table 4: Scalar Higgs-like pairs

projection operators $L_5 = \frac{1}{2}(1 - \tilde{\gamma}_5)$, $R_5 = \frac{1}{2}(1 + \tilde{\gamma}_5)$, giving $R_5(\phi_1 + \phi_2)L_5 = \phi_1 + \phi_2$, $L_5(\phi_1 + \phi_2)R_5 = 0$, $L_5(\phi_1 - \phi_2)R_5 = \phi_1 - \phi_2$, $R_5(\phi_1 - \phi_2)L_5 = 0$. This leads to the gauge-invariant Lagrangian

$$\frac{1}{N_f} \text{tr} \{ [m_U \Psi_R^{U\dagger}(x) [\phi_1(x) + \phi_2(x)] \Psi_L^Q(x) + m_D \Psi_L^{Q\dagger}(x) [\phi_1(x) - \phi_2(x)] \Psi_R^D(x)] P_f \} + \{cc\}, (32)$$

in terms of the scalar fields $\phi_1(x) = \begin{pmatrix} \psi_1^+(x)\phi_1^+ \\ \psi_1^0(x)\phi_1^0 \end{pmatrix}$ and $\phi_2(x) = \begin{pmatrix} \psi_2^+(x)\phi_2^+ \\ \psi_2^0(x)\phi_2^0 \end{pmatrix}$, and quark fields $\Psi_R^U(x) = \sum_\alpha \psi_{UR}^\alpha(x) U_R^\alpha$, $\Psi_R^D(x) = \sum_\alpha \psi_{DR}^\alpha(x) D_R^\alpha$, and $\Psi_L^Q(x) = \sum_\alpha \begin{pmatrix} \psi_{UL}^\alpha(x) U_L^\alpha \\ \psi_{DL}^\alpha(x) D_L^\alpha \end{pmatrix}$, where P_f is a projection operator, α is a spin component, and, with hindsight, we assign the masses

$$m_U = (a + b)/2, \quad m_D = (a - b)/2. \quad (33)$$

Eq. 32's configuration makes manifest the required gauge symmetries: $SU(2)_L$ for a field $\Psi(x)$

$$\Psi(x) \rightarrow e^{i \sum_c \alpha_c(x) I_c} \Psi(x) e^{-i \sum_d \alpha_d(x) I_d} \quad (34)$$

leads to the non-trivial transformations

$$\phi_1(x) - \phi_2(x) \rightarrow e^{i \sum_c \alpha_c(x) I_c} [\phi_1(x) - \phi_2(x)] \quad (35)$$

$$\phi_1(x) + \phi_2(x) \rightarrow [\phi_1(x) + \phi_2(x)] e^{-i \sum_d \alpha_d(x) I_d} \quad (36)$$

$$\Psi_L^Q(x) \rightarrow e^{i \sum_c \alpha_c(x) I_c} \Psi_L^Q(x), \quad (37)$$

and for the $U(1)_Y$ transformation

$$\Psi(x) \rightarrow e^{i \alpha_Y(x) Y} \Psi(x) e^{-i \alpha_Y(x) Y} \quad (38)$$

implies

$$\phi_1(x) - \phi_2(x) \rightarrow e^{i \alpha_Y(x) 1/3} [\phi_1(x) - \phi_2(x)] e^{i \alpha_Y(x) 2/3} \quad (39)$$

$$\phi_1(x) + \phi_2(x) \rightarrow e^{i \alpha_Y(x) 4/3} [\phi_1(x) + \phi_2(x)] e^{-i \alpha_Y(x) 1/3} \quad (40)$$

$$\Psi_L^Q(x) \rightarrow e^{i \alpha_Y(x) 1/3} \Psi_L^Q(x) \quad (41)$$

$$\Psi_R^U(x) \rightarrow e^{i \alpha_Y(x) 4/3} \Psi_R^U(x) \quad (42)$$

$$\Psi_R^D(x) \rightarrow e^{-i \alpha_Y(x) 2/3} \Psi_R^D(x). \quad (43)$$

These relations imply scalar components are connected to the SM Higgs \mathbf{H} through the assignments

$$\mathbf{H}(\mathbf{x}) \sim \phi_1(x) - \phi_2(x) \quad (44)$$

$$\tilde{\mathbf{H}}^\dagger(x) \sim \phi_1(x) + \phi_2(x), \quad (45)$$

where the conjugate representation corresponds to $\tilde{\mathbf{H}}(x) = i I_2 \mathbf{H}^*(x)$, a unitary transformation connects them to their conjugates, e. g. (see Table 4),

$$\phi_1^{+\dagger} + \phi_2^{+\dagger} = -2 I_2 \gamma_2 (\phi_1^0 + \phi_2^0)^* \gamma_2, \quad (46)$$

and the Dirac representation for the γ_μ matrices fixes charge conjugation.

After the Higgs mechanism[25]-[27], only neutral fields survive, and the same basis as Table 4 for the vacuum expectation value is used, leading to the mass Lagrangian

$$H_v = a \phi_1^0 + b \phi_2^0 + a \phi_1^{0\dagger} + b \phi_2^{0\dagger}, \quad (47)$$

This term produces fermion eigenstates and masses from the Yukawa coupling parameters through the relations

$$\begin{aligned} H_v U_M^1 &= m_U U_M^1, & H_v U_M^{c1} &= -m_U U_M^{c1}, \\ H_v D_M^1 &= m_D D_M^1, & H_v D_M^{c1} &= -m_D D_M^{c1}, \end{aligned} \quad (48)$$

where U_M^{c1} , D_M^{c1} correspond to negative-energy solution states (and similarly for opposite spin components.) These states are listed on Table 5 with their quantum numbers;⁶ only two flavors are shown. The role played by m_U , m_D in Eq. 48 confirms

massive quarks	H_v	Q	$\frac{3i}{2}B\gamma^1\gamma^2$
$U_M^1 = \frac{1}{\sqrt{2}}(U_L^1 + U_R^1)$	m_U	$2/3$	$1/2$
$D_M^1 = \frac{1}{\sqrt{2}}(D_L^1 - D_R^1)$	m_D	$-1/3$	$1/2$
$U_M^{c1} = \frac{1}{\sqrt{2}}(U_L^1 - U_R^1)$	$-m_U$	$2/3$	$1/2$
$D_M^{c1} = \frac{1}{\sqrt{2}}(D_L^1 + D_R^1)$	$-m_D$	$-1/3$	$1/2$

Table 5: Massive quarks eigenstates of H_v

their mass interpretation in Eq. 33. In addition, the particular dependence on the a , b parameters implies a flavor-doublet mass hierarchy effect, if they represent a comparable large scale $O(a) \simeq O(b)$. This interpretation is supported by the connections among the Higgs components on Table 4 $\phi_2 = \gamma_5 \phi_1$, and as ϕ_2 can be generated from ϕ_1 by the transformation

$$\phi_2 = -ie^{i\beta\gamma_5} \phi_1 e^{-i\beta\gamma_5} \quad (49)$$

for $\beta = \pi/4$; further on, by a compositeness property, as one may construct the Higgs wave function from the fermions. This is shown in the relations

$$\phi_1^{0\dagger} + \phi_2^{0\dagger} = U_L^1 U_R^{1\dagger} + U_L^2 U_R^{2\dagger} \quad (50)$$

$$\phi_1^{0\dagger} - \phi_2^{0\dagger} = -D_L^1 D_R^{1\dagger} - D_L^2 D_R^{2\dagger}, \quad (51)$$

and the second spin component may be obtained by flipping the spin, for example, $D_L^2 = \frac{3i}{2}B(\gamma_2\gamma_3 - i\gamma_3\gamma_1)D_L^1$.

⁶The fermion states shown can be interpreted as either massive quarks or massive leptons (charged particle and neutrino pairs), according to the choice of the Y operator.

6. Conclusions

This paper presented two related themes: one formal, dealing with translating a previously proposed SM extension to a Lagrangian formalism, and the other phenomenological, dealing with deriving a hierarchy effect from the model. It explained steps aimed at the model's formalization, providing a field-theory formulation; the final objective is to use its restrictions to obtain SM information. Conversely, a field theory can be formulated in this basis, which may provide insight into the symmetries and representations used.

A matrix space is used in which both symmetry generators and fields are formulated. For given dimension, a chosen non-trivial projection operator \mathcal{P}_P constrains the matrix space, determining the symmetry groups, and the arrangement of fermion and boson representations. In particular, spin-1/2, and 0 states are obtained in the fundamental representation of scalar groups and spin-1 states in the adjoint representation. After expressing fields within this basis, a gauge-invariant field theory is constructed, based on the Lorentz and obtained scalar symmetries.

Features obtained from the (5+1)-d extension are formulated through a Lagrangian: the gauge symmetry $SU(2)_L \otimes U(1)_Y$ and global lepton $U_{Le}(1)$ groups with the vector bosons associated to $SU(2)_L$, acting only on the model's predicted representations: left-handed fermions; a scalar doublet associated to a Higgs particle; leading to scalar vector and fermion vertices. Special features emerge in the Lagrangian construction, as the need of a projection operator and Dirac-matrix rules to maintain Lorentz invariance. Within the (7+1)-d case, we showed a pair of Higgs-like scalars induce hierarchy in the masses of flavor-doublet fermions, confirming the model's predictive power.

The paper's SM extension satisfies basic requirement of correct symmetries, including Lorentz and gauge ones, description of SM particles, and field-theory formulation, in addition to its SM prediction provision (the latter two is what the paper deals with.) This supports the view that it is an extension worth considering.

With the Poincaré and SM-gauge symmetric Lagrangian presentation of the model, renormalization and quantization conditions can be applied, leading to a quantum field theory formulation.

A future goal is to apply this framework to supersymmetry. The latter has in common with the extended spin representations classified by a Clifford algebra with Lorentz indexing. This suggests a closer connection between these frameworks. As restrained matrix spaces provide information on fundamental interactions and physical-particle representations, it is worth investigating whether this information can be

obtained within supersymmetry, with the ultimate goal of explaining the origin of interactions.

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